University of California, Berkeley
Physics 105 Fall 2000 Section 1 (*Strovink*)

ASSIGNMENT 12

Reading:
105 Notes 14.6
Hand & Finch 10.1-10.2

1.
Consider a uniform cube of side L. Inside the cube is a scalar field ϕ that satisfies the wave equation with characteristic wavespeed c. At the surfaces of the cube, ϕ is required to vanish.
(**a**)
Show that for this system the total number of modes of vibration corresponding to frequencies between ν and $\nu + d\nu$ is $4\pi L^3\nu^2 d\nu/c^3$, if $\pi c/L \ll d\nu \ll \nu$.
(**b**)
What would the result be for a (two-dimensional) square?
(**c**)
A (one-dimensional) rod?

2. and **3.** (double credit problem)
Consider a homogeneous isotropic *solid* medium, *i.e.* a medium that, unlike a liquid, is able to resist being twisted (it "supports a shear stress"). The Lagrangian density for such a medium is

$$\mathcal{L}' = \frac{1}{2}\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} - \frac{1}{2}\frac{\partial u_i}{\partial x_j}C_{ijkl}\frac{\partial u_k}{\partial x_l}\ ,$$

where summation over repeated indices is (definitely!) implied. In this expression, the field variables are $u_1(x_1, x_2, x_3, t)$, $u_2(x_1, x_2, x_3, t)$, and $u_3(x_1, x_2, x_3, t)$. These describe the (vector) displacement $\mathbf{u}$ of a small element of the solid from its equilibrium position $\mathbf{x}$. (The *strain* is obtained by taking spatial derivatives of $\mathbf{u}$.) The mass density of the solid is ρ, which for small values of $\mathbf{u}$ can be approximated as a constant. C_{ijkl} is the "fourth-rank tensor of elasticity".

Exploiting the homogeneous medium's isotropy, one can show that the most general form for C_{ijkl} is

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\ ,$$

where λ and μ, the so-called "Lamé constants", determine all 81 of its elements. The inverse of the *compression modulus* λ is proportional to the compressibility of the medium, and the inverse of the *shear modulus* μ is proportional to the extent to which the medium can be twisted.

Notice that the Lagrangian density for a solid medium could in principle depend on 19 variables (3 field variables, 3×4 derivatives of 3 field variables with respect to 4 independent variables, and 4 independent variables). In practice, our Lagrangian density has no dependence on the first and last category, so it is a function of only 12 variables.

Use the Euler-Lagrange equations for this Lagrangian density to derive the wave equations for compression waves ($\nabla\times\mathbf{u} = 0$) and for shear waves ($\nabla\cdot\mathbf{u} = 0$) in the solid. Obtain the phase velocity c for both cases, in terms of λ, μ, and ρ. Notice that an earthquake can propagate with more than one velocity!

4.
Consider an infinitely long continuous string in which the tension is τ. A mass M is attached to the string at $x = 0$. If a sinusoidal wave train with velocity ω/k is incident from the left, analyze the reflection and transmission that occur at $x = 0$. Define the reflection coefficient $R \equiv |\mathcal{R}|^2$ and the transmission coefficient $T \equiv |\mathcal{T}|^2$, where $\mathcal{R}$ and $\mathcal{T}$ are the reflected and transmitted amplitude ratios discussed in Lecture Notes section 14.6.

Show that R and T are given by $R = \sin^2\theta$ and $T = \cos^2\theta$, where $\tan\theta = M\omega^2/2k\tau$. [*Hint*: Consider carefully the boundary condition on the derivatives of the wave functions at $x = 0$.]

5.
Hand & Finch, Problem 10.1 (stability in a central force)

6.
Hand & Finch, p. 397, *Question* 6 (upside-down pendulum)

7. and **8.** (double credit problem)
Hand & Finch, Problem 10.9 (a)-(d) *only* (how does a child pump a swing?)